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Reduced 2-to-1 maps and decompositions of graphs with no 2-to-1 cut sets

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Abstract

A graph has an increasing ear decomposition if it can be constructed from a simple closed curve by attaching arcs in stages with the endpoints of each arc attached to different points so that at least one new branch point is formed at each stage. A reduced 2-to-1 map is a 2-to-1 map that does not have a restriction that is 2-to-1. A 2-to-1 cut set of a graph G is a finite subset B such that $G \setminus B$ has at least $2|B|$ components. A graph has an increasing ear decomposition if and only if it does not have a 2-to-1 cut set, and a graph is the image of a reduced 2-to-1 map if and only if it does not have a 2-to-1 cut set.

MSC: 54C10

Keywords: Reduced map; 2-to-1 map; Graph; Increasing ear decomposition

1. Introduction

In a previous paper [2] we showed that a graph is the image of a 2-to-1 map with no 2-to-1 restriction if and only if it does not contain a 2-to-1 cut set. In this paper we will characterize those graphs with no 2-to-1 cut set, a generalization of not having a cut point, by showing that they all have a particular type of open ear decomposition called an increasing ear decomposition. This in the tradition of Whitney [1] who showed that a graph does not have a cut point if and only if it has an open ear decomposition. According to Lovász [3, p. 39]

> Problems involving connectivity between two points are usually settled without difficulty using Menger's theorem. On the other hand, connectivities between more

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that two points are more difficult to handle and are, to a large extent, independent of Menger's theorem. Such problems arise in the study of minimal k-connected graphs, multicommodity flows, safe communication networks, etc. Their solutions are difficult but some typical manipulations with cuts occur repeatedly and these may lead to ideas for a general approach.

Some of the strongest results in the field are structure theorems, which prove that certain classes of graphs can be constructed by repeated application of some simple transformation, e.g., 2-connected graphs by repeatedly attaching "ears".

There are a number of open questions concerning 2-to-1 maps and graphs. For example, Jo Heath has asked if there is a way to determine for a pair of graphs if there is a 2-to-1 map from one to the other. Since there is a reduced 2-to-1 image at the core of every 2-to-1 image, it is hoped that the simple inductive construction we provide for all graphs that are reduced 2-to-1 images might be used to solve more general problems.

2. Preliminaries

A *continuum* is a compact connected topological space. A *graph* is a continuum which is a finite union of arcs with a finite number of points having order greater than two. A *vertex set*, $V(G)$, for a graph G can be any finite subset of G that contains all of the points with order greater than two. Let $E(G)$ represent the set of *edges* of G, that is, the components of $G \setminus V(G)$. A finite subset B of a continuum X is called a *k-to-1 cut set* if $X \setminus B$ has at least $k|B|$ components. A *map* is a continuous function. A *reduced map* between continua is one such that each proper subcontinuum of the range has disconnected preimage. A map is k-to-1 if each point in the image has exactly k points in its preimage. It is easy to see that a k-to-1 map is reduced if and only if it is not k-to-1 when restricted to a proper subcontinuum of the domain. In [2] it is shown that if a graph does not contain a k-to-1 cut set, then it is the image of a nice reduced k-to-1 map from graph. That niceness is described by the following definition. A *routing* of a graph G onto a graph H is a continuous function f that maps $V(G)$ onto $V(H)$, and maps $G \setminus V(G)$ onto $H \setminus V(H)$ sending each edge of G homeomorphically onto an edge of H. The routing f is called a k-routing if $V(G)$ is mapped k-to-1 onto $V(H)$, and at most k edges of G are mapped onto each edge of H. The k-routing f is reduced if it does not have a restriction to a subgraph of G with fewer vertices that is a k-routing. It is easy to see that a k-routing from G onto H is reduced if and only if no proper subcontinuum of H that intersects $V(H)$ has connected preimage.

The relevant results from [2] are summarized below for the case $k = 2$.

Theorem 1 [2, Theorem 3]. *For a graph G the following are equivalent*:

(i) *There is a reduced* 2-*to*-1 *map from a continuum onto G.*
(ii) *There is a reduced* 2-*routing of a tree onto G.*
(iii) *G does not contain a* 2-*to*-1 *cut set.*

Theorem 2 [2, Corollary 1]. *There is a reduced* 2*-routing of a tree* T *onto a graph* G *such that only one edge of* T *maps onto each edge of* G *if and only if* G *does not contain a* 2*-to-*1 *cut set, and* $|E(G)| = 2|V(G)| - 1$.

Theorem 3 [2, Corollary 3]. *A* 2*-routing of a tree* T *onto a graph* G *such that only one edge of* T *maps onto each edge of* G *is reduced if and only if* G *does not contain a* 2*-to-*1 *cut set.*

A simple open path is a collection of edges and vertices whose union is an arc. A simple closed path is a collection of edges and vertices whose union is a simple closed curve. An open ear decomposition starting with P_0 of a graph G is a decomposition $G = P_0 \cup P_1 \cup \cdots \cup P_k$ where P_{i+1} is a simple open path whose end points belong to $P_0 \cup P_1 \cup \cdots \cup P_i$, but whose interior points do not. As mentioned earlier, a graph does not have a cut point if and only if it has an open ear decomposition starting with a simple closed path [1]. An increasing ear decomposition is an open ear decomposition starting with a simple closed path such that for each i, at least one of the end points of P_i has order three in $P_0 \cup P_1 \cup \cdots \cup P_i$. That is, the number of branch points increases at each stage.

3. The decomposition

An increasing ear decomposition is an easy thing to construct for small graphs. However, even in small graphs one can at least begin to see the difficulty of obtaining such a decomposition for the general graph with no 2-to-1 cut set.

Example 1. For each of the graphs below the numbering of the edges indicates an increasing ear decomposition. There are many other ways to decompose each graph, but it is not hard to find a simple closed path in each graph that cannot be the starting path of an increasing ear decomposition of that graph. (See Fig. 1.)

Given a graph with no 2-to-1 cut set, our approach will be to find an increasing ear decomposition in reverse. That is, to identify an edge with an order three end point that, when removed from the graph, leaves a graph that still has no 2-to-1 cut set. Each of the graphs above contains more than one example of such an edge, but there are edges in each graph with order three endpoints that can not be removed without leaving a graph with a

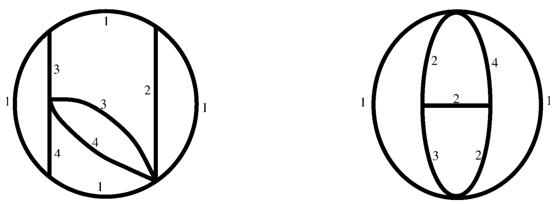

Fig. 1.

2-to-1 cut set. We will show how to pick such an edge using a strongly connected digraph that is related to the reduced 2-routing guaranteed by Theorem 2.

Suppose f is a k-routing of the tree, T, onto the graph, G. For a subset, S, of $V(T)$, let $\langle S\rangle_T$, or just $\langle S\rangle$ when T is understood, represent the unique minimal subgraph of T that contains S. For pairs of vertices $\langle u,v\rangle_T = \langle\{u,v\}\rangle_T$. Define I_f to be a digraph with $V(I_f) = V(G)$ such that for any two vertices a and b of G there is a directed edge (a,b) in I_f if and only if $\langle f^{-1}(a)\rangle_T \cap f^{-1}(b) \neq \emptyset$. By a path in I_f from a to b, it is meant a directed path in I_f from a to b. If a and b are elements of I_f, let the distance from a to b, $d(a,b)$, be the length of the shortest path in I_f from a to b when there is such a path, and let $d(a,b)$ equal infinity otherwise. The length of a path is the number of edges in the path. For each $a \in I_f$, let $R_i(a) = \{x \in V(I_f) \mid d(a,x) \leqslant i\}$, let $R(a)$ be the set of all $x \in V(I_f)$ such that there is a path in I_f from a to x, and let $R_0(a) = \{a\}$. A digraph G is *strongly connected* if for any two vertices a and b there is a directed path in G from a to b and a directed path in G from b to a.

In spite of this somewhat technical definition, the digraph I_f is very easy to produce from the tree and the routing, and is not hard to understand when it is thought of geometrically (see Example 2). The important thing here is that the connectivity of I_f completely captures what it means for a k-routing from a tree to be reduced.

Lemma 1. *If f is a k-routing of a tree, T, onto a graph, G, then f is reduced if and only if I_f is strongly connected.*

Proof. Suppose f is reduced. Let $a \in V(I_f) = V(G)$. If $x \in R_i(a) \setminus R_{i-1}(a)$, then there exists $b \in R_{i-1}(a)$ such that $f^{-1}(x) \cap \langle f^{-1}(b)\rangle \neq \emptyset$. Therefore,

$$\bigl\langle f^{-1}\bigl(R_{i-1}(a)\bigr)\bigr\rangle \cap \bigl\langle f^{-1}(x)\bigr\rangle \neq \emptyset$$

for every $x \in R_i(a)$. Thus,

$$\bigl\langle f^{-1}\bigl(R_{i-1}(a)\bigr)\bigr\rangle \cup \bigcup_{x\in R_i(a)} \bigl\{\bigl\langle f^{-1}(x)\bigr\rangle\bigr\}$$

is connected. Therefore,

$$\bigl\langle f^{-1}\bigl(R_i(a)\bigr)\bigr\rangle = \bigl\langle f^{-1}\bigl(R_{i-1}(a)\bigr)\bigr\rangle \cup \bigcup_{x\in R_i(a)} \bigl\{\bigl\langle f^{-1}(x)\bigr\rangle\bigr\}.$$

So, if $z \in \langle f^{-1}(R_i(a))\rangle \cap V(T)$, and z is not in $f^{-1}(R_i(a))$, then $z \in \langle f^{-1}(x)\rangle$ for some $x \in R_i(a)$, which implies that $(x, f(z)) \in I_f$, and therefore, $z \in f^{-1}(R_{i+1}(a))$. We have shown that $\langle f^{-1}(R_i(a))\rangle \cap V(T) \subseteq f^{-1}(R_{i+1}(a))$ for each $i \geqslant 0$.

It follows that

$$\bigl\langle f^{-1}\bigl(R(a)\bigr)\bigr\rangle \cap V(T) = f^{-1}\bigl(R(a)\bigr).$$

Therefore, the restriction of f to $\langle f^{-1}(R(a))\rangle$ is a k-routing. Since f is reduced, $\langle f^{-1}(R(a))\rangle = T$, and thus, $f^{-1}(R(a)) = V(T)$. Thus, I_f is strongly connected since $R(a) = V(I_f)$ for every $a \in V(I_f)$.

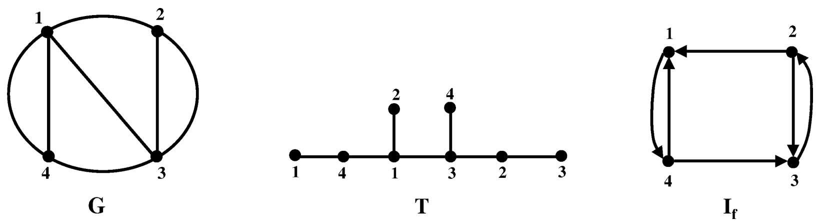

Fig. 2.

Assume I_f is strongly connected. Let K be a connected subgraph of G such that $f^{-1}(K)$ is connected. Suppose $a \in K \cap V(G)$. If $(a,b) \in I_f$, then $\langle f^{-1}(a)\rangle \cap f^{-1}(b) \neq \emptyset$. So $f^{-1}(K) \cap f^{-1}(b) \neq \emptyset$, which implies $b \in K$. Thus, if there is a path in I_f from a to b, then $a \in K$ implies $b \in K$. Therefore, if K intersects $V(G)$, then $V(T) \subseteq f^{-1}(K)$. So, $T = f^{-1}(K)$, and $K = G$. That is, f is reduced. □

Example 2. The graph G is pictured in Fig. 2. The tree T is labeled to indicate a 2-routing f onto G that must be reduced since G does not contain a 2-to-1 cut set. The digraph I_f is strongly connected. (To reduce clutter, I_f is pictured without the edges of the form (v, v).)

Lemma 2. *If the graph G does not contain a 2-to-1 cut set, and $|E(G)| = 2|V(G)| - 1$, then there is an order three vertex v in G, and an edge d incident to v such that $H = G \setminus d$ does not have a 2-to-1 cut set, H is connected, and $|E(H)| = 2|V(H)| - 1$.*

Proof. According to [2, Theorem 5], if G does not have a 2-to-1 cut set, and $|E(G)| = 2|V(G)| - 1$, then G contains two edge disjoint spanning trees T_1 and T_2 such that $G \setminus (T_1 \cup T_2)$ is the interior of a single edge e of G. Let r and s be the endpoints of e, and form a single tree, T, by taking disjoint copies of T_1 and T_2, and adding an edge from the copy of r in the copy of T_1 (call this point r_1) to the copy of s in the copy of T_2. Let f be the natural 2-routing of T onto G which is 1-to-1 on the edges of T. By Theorem 3, this 2-routing is reduced and by Lemma 1, I_f is strongly connected.

To the pair of vertices in T that map onto $x \in V(G)$ assign the names x_1 and x_2 arbitrarily. Suppose the tree T is transformed into another tree, T', by removing a component of $T \setminus \langle x_1, x_2\rangle_T$ that was attached to T at x_i and attaching it at x_j. Let f' be the natural 2-routing of T' onto G.

Claim 1. *Since I_f is strongly connected, $I_{f'}$ is strongly connected.*

This follows immediately from Theorem 3 and Lemma 1.

Claim 2. *If a, b, and v are vertices of I_f with $v \neq x$ such that there is a directed path in $I_f \setminus \{v\}$ from a to b, then there is a directed path from a to b in $I_{f'} \setminus \{v\}$.*

Proof. To prove this, notice that if $(u, w) \in I_f$ but $(u, w) \notin I_{f'}$, then $(u, x) \in I_{f'}$, and $(x, w) \in I_{f'}$. So, if P is a path in $I_f \setminus \{v\}$ from a to b, replace each edge (u, w) of P that is not also an edge in $I_{f'}$ with $(u, x) \cup (x, w)$. The result is a path in $I_{f'} \setminus \{v\}$ from a to b. □

Claim 3. *Suppose $v \in V(I_f)$ such that $v \neq x$ and v, I_f, and T have the property that for any a and b in $V(I_f) \setminus \{v\}$ such $b_j \in \langle a_i, r_1\rangle_T$ for any values of i and j, then there is a directed path in $I_f \setminus \{v\}$ from a to b. Then v, $I_{f'}$ and T' have the same property. That is, for any vertices a' and b' of $V(I_{f'}) \setminus \{v\}$ such that $b'_j \in \langle a'_i, r_1\rangle_{T'}$ for any values of i and j, then there is a path in $I_{f'} \setminus \{v\}$ from a' to b'.*

Proof. To prove this, notice that if $b_j \in \langle a_i, r_1\rangle_{T'}$ and $b_j \in \langle a_i, r_1\rangle_T$, then the path from a to b in $I_{f'}$ exists by Claim 2. So, without loss of generality, assume that $b_1 \in \langle a_1, r_1\rangle_{T'}$, and $b_j \notin \langle a_i, r_1\rangle_T$ for any choice of i and j. Let C be the component of $T \setminus \langle x_1, x_2\rangle_T$ that is moved to form T', and assume it is moved from x_1 to x_2. Obviously, $a_1 \in C$, but $b_1 \notin C$ and $r_1 \notin C$. So, in this case, $x_1 \in \langle a_1, r_1\rangle_T$. So, there is a path in $I_f \setminus \{v\}$ from a to x, and therefore, by Claim 2, there is a path P in $I_{f'} \setminus \{v\}$ from a to x. Note also that in this case $b_1 \in \langle x_1, x_2\rangle_{T'}$. Therefore, $I_{f'} \setminus \{v\}$ contains the path $P \cup (x, b)$ which goes from a to b. This concludes the proof of the claim. □

Let B be the set of all vertices in G with order greater than three. For each $x \in B$ note that $T \setminus \langle x_1, x_2\rangle_T$ has at least two components. Therefore, if x_1 or x_2 is an end point of T, it is possible to transform T in the manner described above so that in the new tree neither x_1 nor x_2 is an end point. Suppose T' is obtained from T by making a finite number of such transformations so that in T' there is no $x \in B$ such that either x_1 or x_2 is an end point. Let f' be the natural 2-routing from T' onto G.

We are now ready to choose the vertex v. Note that since $|E(G)| = 2|V(G)| - 1$, there must be at least two vertices in $V(G) \setminus B$. Let v be the element of $V(G) \setminus B$ for which the distance from r to v in $I_{f'}$ is maximized over all elements of $V(G) \setminus B$.

Applying the three claims above for each of the transformations from T to T', we see that T', f' and $I_{f'}$ have the following properties:

(*) $I_{f'}$ is strongly connected.

(**) For each $a \in V(G) \setminus \{v\}$ there is a path in $I_{f'} \setminus \{v\}$ from a to r.
This follows from the second claim since if $a \in V(G) \setminus \{v\}$, then $r_1 \in \langle a_1, a_2\rangle_T$. So, I_f contains the directed edge (a, r) which is a path in I_f from a to r.

(***) If a and b are in $V(G) \setminus \{v\}$, and there are integers i and j such that $b_j \in \langle a_i, r_1\rangle$, then $I_{f'} \setminus \{v\}$ contains a path from a to b.

This follows from the third claim, since if a and b are in $V(G) \setminus \{v\}$ such that for some i and j we have $b_j \in \langle a_i, r_1\rangle_T$, then $b_j \in \langle a_1, a_2\rangle_T$. So, I_f contains (a, b) which is a path in I_f from a to b that does not contain v.

We claim that $I_{f'} \setminus \{v\}$ is strongly connected. To see this, let a and b be arbitrary elements of $V(I_{f'}) \setminus \{v\}$. From (**) it follows that there is a path, P_1, in $I_{f'} \setminus \{v\}$ from a to r. If $b \in V(I_{f'}) \setminus B$, then there must be a path, P_2, in $I_{f'} \setminus \{v\}$ from r to b because the distance in $I_{f'}$ from r to v is greater than the distance from r to b. In this case the path $P_1 \cup P_2$ is a path in $I_{f'} \setminus \{v\}$ from a to b.

So assume $b \in B$. Since neither b_1 nor b_2 is an end point of T', there are end points e_1 and e_1' such that $b_1 \in \langle e_1, r_1\rangle_{T'}$ and $b_2 \in \langle e_1', r_1\rangle_{T'}$. Since the reduced map f' cannot send two endpoints to the same vertex in G, at least one of e_1 and e_1' is not mapped by f' onto v. Suppose $f(e_1) \neq v$. Let $f(e_1) = e$. Since e_1 is an endpoint and $e \notin B$, then according to the argument at the end of the previous paragraph there is a path P_2 in $I_{f'} \setminus \{v\}$ from r to e. From (***) it follows that there is a path P_3 in $I_{f'} \setminus \{v\}$ from e to b. Therefore, the path $P_1 \cup P_2 \cup P_3$ is a path in $I_{f'} \setminus \{v\}$ from a to b. It follows that $I_{f'} \setminus \{v\}$ is strongly connected.

Since v has order three in G, either v_1 or v_2 must be an end point of T'. Assume v_1 is an end point of T'. Note that v_2 must have order 2 in T'. Form the tree T'' by removing the edge from T' that contains v_1, and by no longer considering the order two point v_2 a vertex. Let d be the image under f' of the edge with end point v_1. If f'' is the obvious 2-routing of T'' onto $H = G \setminus d$, then $I_{f''} = I_{f'} \setminus \{v\}$. Thus, $I_{f''}$ is strongly connected, and therefore, H does not contain a 2-to-1 cut set. H is connected since T'' is connected, and H has one fewer vertex than G and two fewer edges, so $|E(H)| = 2|V(H)| - 1$. $\square$

Example 3. For the graph G also pictured in Example 2, a new 2-routing from a tree that is based on two edge disjoint spanning trees of the graph G is indicated. The new digraph I_f is very different from the one in the previous example. It is clear that eliminating the vertex numbered 2 leaves a digraph that is still strongly connected, while no vertex could be eliminated from the digraph in Example 2 that would leave a strongly connected digraph. See Fig. 3.

The resulting digraph, $I_{f'}$, corresponds to the 2-routing from a tree, T', onto the graph, G', indicated below. See Fig. 4.

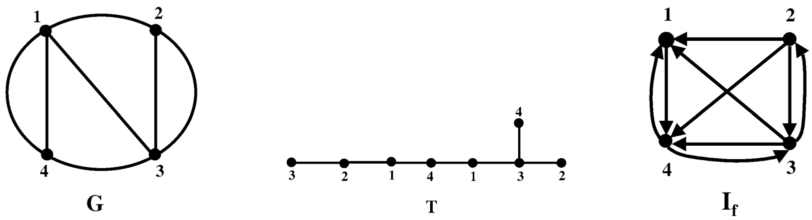

Fig. 3.

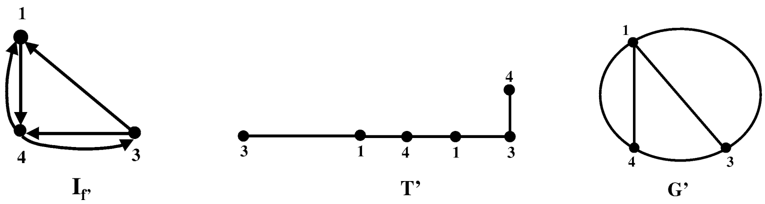

Fig. 4.

The main theorem follows easily from the previous lemma and a few more results from [2].

Theorem 4. *For a graph G, the following are equivalent.*

(1) *There is a reduced* 2-*to*-1 *map from a continuum onto* G.
(2) *There is a reduced* 2-*routing from a tree onto* G.
(3) G *does not have a* 2-*to*-1 *cut set.*
(4) G *has an increasing ear decomposition.*

Proof. The equivalence of (1), (2), and (3) is established in [2, Theorem 3], and the fact that (4) implies (3) is [2, Theorem 2]. We have left to show only that if G does not have a 2-to-1 cut set, then G does have an increasing ear decomposition.

It is shown in the proof of [2, Lemma 7] that if a graph G does not have a 2-to-1 cut set, then there is a graph G' that contains G such that $V(G) = V(G')$, $|E(G')| = 2|V(G')| - 1$, G' does not have a 2-to-1 cut set, and the edges of $E(G') \setminus E(G)$ duplicate edges of G. That is, if e' is an edge in $E(G') \setminus E(G)$ from vertex a to vertex b, then $E(G)$ also contains an edge from a to b. It is easy to see that if G' has an increasing ear decomposition, then each duplicate edge in $E(G') \setminus E(G)$ together with its end points must be path in the decomposition. Leaving out those paths in the decomposition will result in an increasing ear decomposition of G.

The theorem follows by induction using Lemma 2, and assuming $|E(G)| = 2|V(G)| - 1$. □

As was seen in an earlier example, there can be simple closed paths in a graph with no 2-to-1 cut set that cannot be the starting path, and there can be order three vertices that cannot be the end point of a final path of any increasing ear decomposition of the graph. However, we will use our characterization to show that, for any edge of a graph with an no 2-to-1 cut set, there is an ear decomposition whose starting simple closed path contains that edge. One consequence is that for any edge there is an ear decomposition that does not contain that edge in its final path.

Theorem 5. *If the graph G has an increasing ear decomposition, and e is any edge of G, then there is an increasing ear decomposition of G whose starting simple closed path contains e.*

Proof. Let e' be an arc contained in e such that e' contains one endpoint of e but not the other. Let G' be the graph obtained by taking two disjoint copies of G and identifying the two copies of e', and then removing the interior of e'. Refer to the two identical halves of this graph as G_1 and G_2. It is an easy exercise to show that G' does not contain a 2-to-1 cut set. Therefore, G' has an increasing ear decomposition. If that decomposition begins entirely in say G_1, then the sequence of edges that are added to the decomposition that intersect G_2 will correspond to an increasing ear decomposition of G that contains e in its initial loop. If the decomposition of G' begins with a loop that contains the remaining part of the edge e in G_1 and its copy in G_2, then the part of that loop that is in G_1 can be closed off in G by putting e' back in. The remaining edges added to the ear decomposition of G' will be contained either in G_1 or G_2. Those contained in G_1 when added to the initial loop described above correspond to an increasing ear decomposition of G that contains e in its initial loop. □

The following is a simple application of the ear decomposition in an induction argument.

Theorem 6. *Suppose the graph G does not have a* 2-*to*-1 *cut set, and v_1 and v_2 are two points with order two in G such that $\{v_1, v_2\}$ does not separate G. If G' is the graph obtained by identifying v_1 with v_2, then G' does not have a* 2-*to*-1 *cut set.*

Proof. The theorem is vacuously true if G has only one edge, that is, if G is a simple closed curve. Assume the theorem is true for all graphs with i edges, and that G has $i+1$ edges. Let $e = (a, b)$ be the final edge of an increasing ear decomposition of G. At most one of $\{v_1, v_2\}$ can be an element of e. Assume v_1 is in e. Then, instead of adding e in the last stage of the ear decomposition, add two new edges (a, v_2) and (v_2, b) to get an increasing ear decomposition of G'. If neither v_1 nor v_2 is in e, then identify v_1 and v_2 in $G \setminus e$, and, according to the inductive hypothesis, this graph does not have a 2-to-1 cut set. Adding e to this graph produces G' which also does not have a 2-to-1 cut set by [2, Theorem 2]. □

4. Generalization

The main results in the present work are only for graphs with no 2-to-1 cut set, and yet many of the results on which the present work are based are results that have been shown for all positive integers. For example, [2, Theorem 2] says:

> *Suppose the connected graph G does not contain a k-to-1 cut set, and the graph G' is obtained from G by adding an edge with ends attached to two different points in G at least one of which has order in G less that or equal to k. Then G' does not have a k-to-1 cut set.*

This raises the possibility of a natural generalization of the increasing ear decomposition that would characterize all graphs with no k-to-1 cut set. However, consider the graph pictured below. See Fig. 5.

There is no 3-to-1 cut set in this graph, but each vertex has order five. There is no ear decomposition where each added path has at least one endpoint with order four. There may

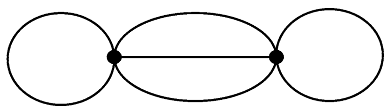

Fig. 5.

be a natural way to construct a graph with no k-to-1 cut set from a sequence of graphs with no k-to-1 cut set, and such a construction might be useful, since for every graph there is a k for which the graph does not contain a k-to-1 cut set. However, it does not appear that it can be done with a decomposition, that is, using subgraphs of the original graph as in the case for $k = 2$.

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